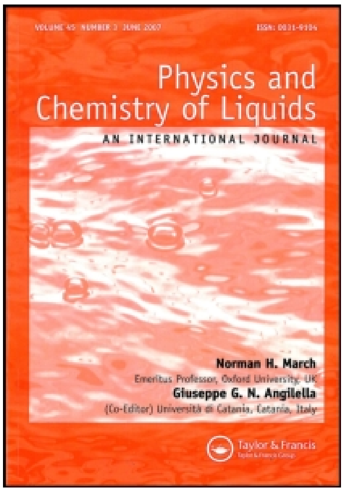

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THE STATISTICAL DISTRIBUTION FUNCTION FOR AN ANYON LIQUID

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The distribution function for fermions has been derived from collision theory by using the detailed balance hypothesis. Its form

$$f(\varepsilon) = \{\exp[\beta(\varepsilon - \mu)] + 1 - 2/n^2\}^{-1}$$

goes to the correct boson ($n = 1$) and fermion ($n = \infty$) limits. Further the equations for the density matrix have been derived from the Hamiltonian. The zero order approximation gives a distribution having the same form as that derived by the detailed balance arguments.

KEY WORDS: Detailed balance, momentum distribution.

1 BACKGROUND

Particles with fractional statistics, termed anyons by Wilczek[1], are of considerable current interest in connection with (a) the fractional quantum Hall effect, (b) the phase diagram of two-dimensional Wigner crystals in strong magnetic fields[2] and (c) high temperature superconductivity[3].

Though treatment of thermodynamics of an anyon gas have been given[4,5], we are not aware of a calculation of the statistical distribution function $f(\varepsilon)$ for states of energy ε of such a gas. The purpose of this paper is two-fold: (a) To derive $f(\varepsilon)$ using the detailed balance hypothesis (Section 2) and; (b) To derive the thermodynamic two-particle density matrix equation for grand canonical ensembles and use approximate solutions to compare the resulting $f(\varepsilon)$ with that of Section 2.

2 COLLISION THEORY

Let us start out from the treatment of collisions in a gas of Fermions following for example, Ma.[6] Suppose we have the collision $1 + 2 \leftrightarrow 3 + 4$, i.e. the states 1,2 interact to change to states 3,4. We note that collisions are necessary to allow thermodynamic equilibrium to be attained, even in an ideal gas. Then following Ma, this reaction has the rate

$$f1f2(1 - f3)(1 - f4)R$$

where $1 - f3$ and $1 - f4$ are the probabilities that there are no particles in the states 3 and 4. These factors must be present for Fermions because if the states 3,4 are occupied then the reaction cannot occur. Now one invokes the fact that this rate must equal that in the reverse direction, i.e.

$$f1f2(1 - f3)(1 - f4)R = f3f4(1 - f1)(1 - f2)R' \tag{1}$$

Ma next uses time reversal symmetry of quantum mechanics to show that $R = R'$ and the resulting equation leads to a solution of the form

$$f(\varepsilon)/(1 - f(\varepsilon)) = \exp(\beta(\mu - \varepsilon)) \tag{2}$$

where μ must be determined by the normalization condition for the particle number N.

We next note that for Bosons the $-$ signs in the factors $(1 - f3)$ and $(1 - f4)$ in Eq. (1) must be replaced by the $+$ signs in order to get the correct distribution: it is as though if states 3 and 4 are occupied, other states would be more inclined to go to them.

Turning to fractional statistics, one finds that "interchange" of two particles introduces a phase factor to the wave function of the form $\exp(+i\pi/n)$ the sign depending on the path followed in transforming the position variables. Thus assuming that detailed balance holds for anyons one has to modify the factor which takes into account the occupancy of the final states 3 and 4. Thus the factor $(1 - f)$ for fermions must become $(1 - a(n)f)$ where the factor $a(n)$ will be taken to have the property,

$$a(\infty) = 1, \qquad a(1) = -1 \tag{3}$$

$$-1 \Leftarrow a(n) \Leftarrow 1 \tag{4}$$

Obviously the "boundary conditions" specified in Eq. (3) for $n = \infty$ and 1 ensure the correct Fermi-Dirac and Bose-Einstein limits.

Then, whereas Ma rewrites Eq. (2) for Bosons and Fermions as

$$f(\varepsilon)\{1 + f(\varepsilon)\}^{-1} = \exp\{(\beta(\mu - \varepsilon)\} \tag{5}$$

one now has for anyons

$$f(\varepsilon)\{1 - a(n)f(\varepsilon)\}^{-1} = \exp\{(\beta(\mu - \varepsilon)\} \tag{6}$$

or

$$f(\varepsilon) = 1/[\exp\{\beta(\varepsilon - \mu)\} + a(n)] \tag{7}$$

Eq. (7) represents the present proposal for the shape of the statistical distribution function for anyons.

The argument of this section does not contain within itself a procedure for determining the anyon shape factor. We can determine however the form of $a(n)$ by imposing that $f(\varepsilon)$ should result in the correct virial coefficient B as calculated by Arovas *et al.*,[7] assuming a proper choice of energy levels ε and density of states.

Taking $\varepsilon = k^2/2$, a calculation of the virial coefficient in the limit of large T ($\beta \to 0$) results in

$$B = \lambda_T^2 a(n)/4,$$

where λ_T is the thermal wavelength $\lambda_T^2 = 2\pi h^2/\text{MkT}$. Introducing the form of the virial coefficient,

$$B = -\lambda_T^2/4 + (1 + 1/n)\lambda_T^2 - (1 + 1/n)^2\lambda_T^2/2,$$

we conclude that

$$a(n) = 1 - 2n^{-2} \tag{8}$$

Finally by combining Eqs. (7) and (8) one has the result

$$f(\varepsilon) = 1/[\exp\{\beta(\varepsilon - \mu\} + 1 - 2n^{-2}] \tag{9}$$

One notes immediately that with the special form (9) arising from the shape of $f(\varepsilon)$ in Eq. (7), the particular case $n = 2^{1/2}$ leads precisely to Boltzmann statistics. Figure 1 shows the distribution for $n^{-1} = 1$, 0.75, 0.5, 0.25 and 0. In each case the value of μ was chosen by normalizing the distribution to unity and $\beta = 5$.

3 DERIVATION OF THE DENSITY MATRIX FROM THE FREE ANYON HAMILTONIAN

We next proceed to the derivation of the density matrix by using the Hamiltonian of the anyon gas and transformations similar to those used by Mori:[8]

$$H = [1/2m] \int d^2\mathbf{r}\psi^+(\mathbf{r})[\mathbf{p} - \mathbf{A}(\mathbf{r})]^2\psi(\mathbf{r}) \tag{10}$$

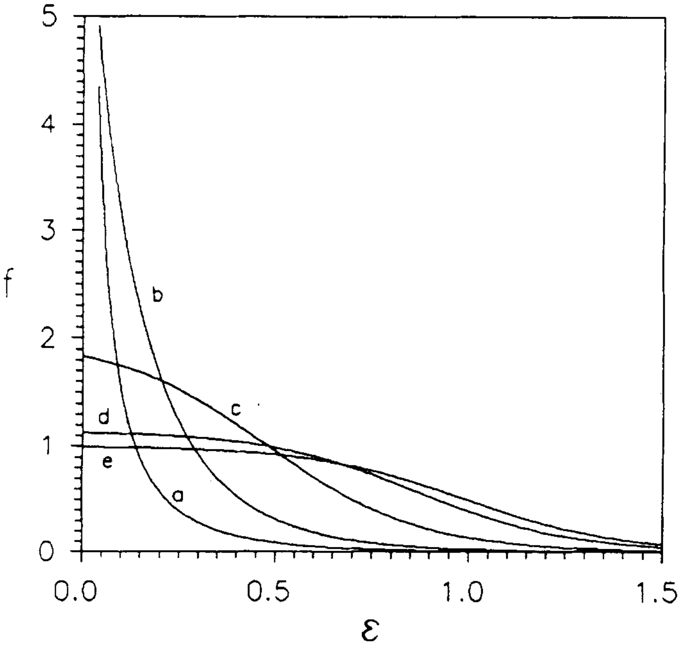

Figure 1 The distribution function $f(\varepsilon)$ given by Eq. (9) for the following values of $1/n = 1$ (line a), 0.75 (b), 0.5 (c), 0.25 (d) and 0 (e) for $\beta = 5$.

$\psi(\mathbf{r})$, $\psi^+(\mathbf{r})$ are the fermion field operators, $\mathbf{A}(\mathbf{r})$ is the vector potential:

$$\mathbf{A}(\mathbf{r}) = \int d^2\mathbf{r}'\mathbf{K}(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \tag{11}$$

where

$$\mathbf{K}(\mathbf{r}) = [1/n]\mathbf{grad}[\theta(\mathbf{r})]$$

and $\theta(\mathbf{r})$ is the azimuthal angle of the two-dimensional vector $\mathbf{r}$ when $r > a$ and 0 otherwise. After integration the limit of a is taken to zero.

We use the operator

$$U(\mathbf{r}) = \exp\left\{[i/n]\int d^2\mathbf{r}'\theta(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\right\} \tag{12}$$

Note that if $U(\mathbf{r}')$ is obtained from $U(\mathbf{r})$ after rotation by 2π about the z axis then $\theta(\mathbf{r}') = \theta(\mathbf{r}) + 2\pi$ and $U(\mathbf{r}') = \exp\{i2\pi N/n\}U(\mathbf{r})$, where $N = \int d^2\mathbf{r}\psi^+(\mathbf{r})\psi(\mathbf{r})$.

Following Mori,[8] we define the anyon field operators

$$\tilde{\psi}^+(\mathbf{r}) = \psi^+(\mathbf{r})U(\mathbf{r}), \qquad \tilde{\psi}(\mathbf{r}) = U^+(\mathbf{r})\psi(\mathbf{r}) \tag{13}$$

Then, by using standard methods of quantum mechanics the following equalities can be derived,

$$U^{+}(\mathbf{r}')\psi^{+}(\mathbf{r})U(\mathbf{r}') = \exp\{[-i/n]\theta(\mathbf{r}' - \mathbf{r})\}\psi^{+}(\mathbf{r}) \tag{14a}$$

$$U^{+}(\mathbf{r}')\psi(\mathbf{r})U(\mathbf{r}') = \exp\{(i/n)\theta(\mathbf{r}' - \mathbf{r})\}\psi(\mathbf{r}) \tag{14b}$$

and by explicit calculation the relations concerning the interchange of the field operators follow:

$$\tilde{\psi}^{+}(\mathbf{r})\tilde{\psi}^{+}(\mathbf{r}') = -\exp\{i\Delta\theta(\mathbf{r} - \mathbf{r}')/n\}\tilde{\psi}^{+}(\mathbf{r}')\tilde{\psi}^{+}(\mathbf{r}) \tag{15a}$$

$$\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r}') = -\exp\{i\Delta\theta(\mathbf{r} - \mathbf{r}')/n\}\tilde{\psi}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \tag{15b}$$

$$\tilde{\psi}(\mathbf{r})\tilde{\psi}^{+}(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - \exp\{[-i\Delta\theta(\mathbf{r} - \mathbf{r}')/n\}\tilde{\psi}^{+}(\mathbf{r}')\tilde{\psi}(\mathbf{r}) \tag{15c}$$

where

$$\Delta\theta(r - r') = \theta(\mathbf{r} - \mathbf{r}') - \theta(\mathbf{r}' - \mathbf{r}) \tag{16a}$$

and therefore

$$\Delta\theta(x, y) = -\pi\,\mathrm{sgn}(y) \tag{17b}$$

The above relations coincide with those of Mori when the relative position $\mathbf{r} - \mathbf{r}'$ corresponds to $z - z'$ in the upper half of the complex plane, whereas the opposite phase factor appears for $z - z'$ in the lower half plane.

The Hamiltonian expressed in terms of these operators is reduced to a free anyon Hamiltonian:

$$H = [1/2m]\int d^2\mathbf{r}\tilde{\psi}^{+}(\mathbf{r})[(\mathbf{p})]^2\tilde{\psi}(\mathbf{r}) \tag{17}$$

For calculating the thermal average of the one particle density matrix, $\rho(\mathbf{r}, \mathbf{r}')$,

$$\langle\rho(\mathbf{r}, \mathbf{r}')\rangle_T = \mathrm{Tr}[\exp(-\beta(H - \mu N))\tilde{\psi}^{+}(\mathbf{r})\tilde{\psi}(\mathbf{r}')]/\mathrm{Tr}[\exp(-\beta(H - \mu N)] \tag{18a}$$

we prove first that it is a function only of the relative coordinates. This is a consequence of fact that the Hamiltonian in its initial as well as its transformed form is invariant under translations and

$$\mathrm{Tr}[\exp(-\beta(H - \mu N))\rho(\mathbf{r} + \mathbf{a}, \mathbf{r} + \mathbf{a}')] = \mathrm{Tr}[\exp(-\beta(H - \mu N))\hat{T}_a\rho(r, r')\hat{T}_a^{-1}] \tag{18b}$$

By transferring the operator $\hat{T}_a^{-1}$ on the left of the Trace argument, the operator action is shifted to the Hamiltonian which is invariant under translations. Hence our statement that the density matrix depends only on the relative coordinate is proved. We next take the thermal average of both sides of (15c) and transfer all creation operators to the left by using the invariance of the trace under cyclic permutations. We get

$$\exp[i\Delta\theta(\mathbf{r}-\mathbf{r}')/n]\langle\tilde{\psi}^{+}(\mathbf{r})\tilde{\psi}(\mathbf{r}')\rangle_T + \langle\tilde{\psi}^{+}(\mathbf{r};\beta)\tilde{\psi}(\mathbf{r}')\rangle_T = \delta(\mathbf{r}-\mathbf{r}') \tag{19}$$

where

$$\tilde{\psi}^{+}(\mathbf{r};\beta) = \exp(\beta(H-\mu N))\tilde{\psi}^{+}(\mathbf{r})\exp(-\beta(H-\mu N)) \tag{20}$$

By using standard methods of quantum mechanics we find

$$\tilde{\psi}^{+}(\mathbf{r};\beta) = \exp(\beta(-\nabla^2/2M-\mu))\psi^{+}(\mathbf{r}) \tag{21}$$

and therefore

$$\langle\tilde{\psi}^{+}(\mathbf{r};\beta)\tilde{\psi}(\mathbf{r}')\rangle_T = \exp(\beta(-\nabla^2/2M-\mu))\langle\psi^{+}(\mathbf{r})\psi(\mathbf{r}')\rangle_T \tag{22}$$

Then after introducing this expression in (19) we find,

$$\{\exp(\beta(-\nabla^2/2M-\mu)) + \exp(in^{-1}\Delta\theta(\mathbf{r}-\mathbf{r}'))\}\langle\rho(\mathbf{r},\mathbf{r}')\rangle_T = \delta(\mathbf{r}-\mathbf{r}') \tag{23}$$

Further simplification is obtained by introducing the center of mass **R** and relative position $\mathbf{s}=\mathbf{r}-\mathbf{r}'$ variables. Using the translation invariance of $\langle\rho\rangle$ we obtain the equation for the free anyon single particle density,

$$(A+B)\rho(\mathbf{r}) = \delta(\mathbf{r}) \tag{24a}$$

where

$$A = \exp(\beta(-\nabla^2/2M-\mu)) + \cos(\pi/n) \tag{24b}$$

and

$$B = -i\,\mathrm{sgn}(y)\sin(\pi/n). \tag{24c}$$

Thus both operators are invariant under translations in the x direction while A is also invariant in the y direction. At first glance it seems strange that the x and y directions are not equivalent. This is not surprising for equations where magnetic interactions are present. A classical example is the choice of the Landau gauge for A in dealing with the electron in a homogenous magnetic field.

In the following we shall proceed to the derivation of approximate results. By considering B as the perturbation term, the solution of (24) is of the form

$$\rho = \sum_n \rho^n, \qquad \rho^n = A^{-1}B\rho^{n-1} \tag{25a}$$

with

$$\rho^0(r) = [\cos(\pi/n + \exp(\beta(k^2/2M - \mu))]^{-1} \exp(ikr)/(L\pi^{1/2}) \tag{25b}$$

where L is the maximum possible distance between two anyons. Thus in terms of formal occupation numbers η_k we have

$$\langle n_k \rangle_T = [\cos(\pi/n + \exp(\beta(k^2/2M - \mu))]^{-1} \tag{26}$$

In this approximation, the occupation numbers have the same form as the ones obtained by using detailed balance considerations. However the virial coefficient calculated by using the zero order approximation ρ^0 is $\lambda_T^2 \cos(\pi/n)/4$, i.e. it differs from that derived by Arovas *et al.*[7] This is due to the fact that in our zero order approximation the nonequivalence of the x and y directions is absent. Hence it is a limited approximation. It is also to be reminded that our thermodynamic quantities refer to a grand canonical ensemble, whereas Arovas *et al.* refer to the case of a canonical ensemble of two particles.

Finally we conclude that higher order approximations to Eq. (24) are needed, where the asymmetry between the x and y axis will manifest its implications.

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